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Hard Metrics From Cayley Graphs Of Abelian Groups

Ilan Newman Yuri Rabinovich*

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Abstract: Hard metrics are the class of extremal metrics with respect to embedding into Euclidean spaces; they incur $\Omega(\log n)$ multiplicative distortion, which is as large as it can possibly get for any metric space of size n. Besides being very interesting objects akin to expanders and good error-correcting codes, and having a rich structure, such metrics are important for obtaining lower bounds in combinatorial optimization, e. g., on the value of MinCut/MaxFlow ratio for multicommodity flows.

For more than a decade, a single family of hard metrics was known (Linial, London, Rabinovich (Combinatorica 1995) and Aumann, Rabani (SICOMP 1998)). Recently, a different family was found by Khot and Naor (FOCS 2005).

In this paper we present a general method of constructing hard metrics. Our results extend to embeddings into negative type metric spaces and into ℓ_1.

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1 Introduction

A famous theorem of Bourgain [4] states that every finite metric space $V = (X,d)$ of size $n = |X|$ can be embedded into a Euclidean space with multiplicative distortion at most $O(\log n)$. We call a metric space V *hard* with respect to ℓ_2 if any embedding of V into a Euclidean space (of any dimension) has a multiplicative distortion $\Omega(\log n)$. Similarly, we call a metric space V *hard* with respect to $\mathcal{M}$, where $\mathcal{M}$ is a class of metric spaces (e. g., $\mathcal{M} = \ell_p$ or NEG), if any embedding of V into $\mathcal{M}$ has multiplicative

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distortion $\Omega(\log n)$. When we use the term *hard* without specifying $\mathcal{M}$, we always mean with respect to $\mathcal{M} = \ell_2$.

When studying a special class of metric spaces, perhaps the most natural first question is whether this class contains hard metrics with respect to ℓ_2. Many fundamental results in the modern theory of finite metric spaces may be viewed as a negative answer to this question for some special important class of metrics. E. g., Arora et al. [2] (improving on Chawla et al. [5]) show this for Negative Type metrics, Rao [14] for planar metrics, and Gupta et al. [7] for doubling metrics. For a long time (since Linial, London and Rabinovich [11] and Rabani and Aumann [3]), the only known family of hard metrics was, essentially, the shortest-path metric of constant-degree expander graphs. It was even suggested that in some vague sense these are essentially the only hard metrics. Recently, however, Khot and Naor [10] constructed a different family of hard metrics by considering certain quotient spaces of $\mathbb{Z}_2^n$ equipped with the Hamming distance.

The starting point of the current research was a plausible conjecture that a *circular* metric cannot be hard, where by circular we mean a metric on the underlying space $\mathbb{Z}_n$, such that $d(a,b)$ depends solely on $((a-b) \mod n)$. Rather surprisingly, the conjecture turns out to be false, and, moreover, it fails not only for $\mathbb{Z}_n$, but for *any* Abelian group H. More precisely, it is always possible to choose a set A of generators for H, so that the shortest-path metric of the corresponding Cayley graph $G(H,A)$ is hard with respect to ℓ_2, ℓ_1 and NEG. In the special case of $\mathbb{Z}_2^n$, good sets of generators are closely related to error-correcting codes of constant rate and linear distance.

Our construction is both simple to describe and easy to analyze. It differs from that of [11, 3], as the degree of such Cayley graphs is necessarily not bounded. Moreover, the construction of [10], despite very different description and analysis, can be shown to produce the same metric space as does our construction in the special case of $\mathbb{Z}_2^n$.

Note: Although in what follows we restrict the discussion to Euclidean spaces, the same method can be used to show the hardness of the metrics that we construct also with respect to the much richer class NEG of "negative type metrics" and consequently to ℓ_1.

2 Definitions

Let (X,d) be a metric space which one wants to embed into another metric space $A = (H,\nu)$. The *multiplicative distortion*, or simply the *distortion* of embedding (X,d) into A is defined as

$$c_A(d) \;=\; \mathrm{dist}(d \hookrightarrow A) \;=\; \min_{\phi: X\to H} \max_{x,y\in X} \frac{\nu(\phi(x),\phi(y))}{d(x,y)} \cdot \max_{x,y\in X} \frac{d(x,y)}{\nu(\phi(x),\phi(y))} \,.$$

We use the terms Euclidean metrics and ℓ_2-metrics interchangeably.

We say that a metric space $V = (X,d)$ is of *negative type* if there is a map f from X to an Euclidean space such that for all $x,y \in X$ we have $d(x,y) = \|f(x) - f(y)\|_2^2$. (So all triangles spanned by the image of X are acute.) NEG denotes the class of negative type metrics.

It is well known that ℓ_1 metrics are of negative type (cf. [6], Part 1, Chapter 6.1 for a comprehensive discussion and historical notes).

3 Abelian Groups

Let $G = (V,E)$ be a d-regular connected graph on n vertices, and let μ_G be its shortest-path metric. For $\Delta : V(G) \times V(G) \to \mathbb{R}^+$, consider the following projective quadratic form, often called a *Poincaré form*:

$$F(\Delta) \;=\; \frac{\sum_{(i,j)\in E(G)} \Delta^2(i,j)}{\sum_{i<j\in V(G)} \Delta^2(i,j)} \tag{3.1}$$

To obtain distortion lower bounds on μ_G, we use the standard (dual) method of comparing Poincaré forms (see, e. g., [11, 13]). Our first step is to get a general lower bound on distortion of embedding μ_G into an Euclidean space.

By the definition above,

$$F(\mu_G) \;=\; \frac{|E|}{\binom{n}{2}\operatorname{avg}(\mu_G^2)}\,,$$

where $\operatorname{avg}(\mu_G^2)$ is the average value of $\mu_G^2(i,j)$ over all pairs (i,j) of distinct vertices of G.

Consider now a Euclidean metric on $V(G)$, $\delta \in \ell_2$, namely, a metric of the form

$$\delta(i,j) \;=\; \|x^i - x^j\|_2\,, \quad \{x^i\}_{i\in V(G)} \subset \mathbb{R}^m\,.$$

If $F(\delta)$ is much larger than $F(\mu_G)$ for *every* such δ, one immediately concludes that any such δ must significantly distort μ_G. Formally,

Proposition 3.1.

$$\operatorname{dist}^2(\mu_G \hookrightarrow \ell_2) \;\geq\; \min_{\delta\in\ell_2} F(\delta)/F(\mu_G)\,.$$

By a standard argument (see e. g., [13], Sect. 15.5), the minimum of $F(\delta)$ over all such δ is precisely γ_G/n, where γ_G is the *spectral gap* of G, that is, $(d-\lambda_G)$ where λ_G is the second largest eigenvalue of the adjacency matrix of G. Thus, Proposition 3.1 implies,

Proposition 3.2.

$$\operatorname{dist}^2(\mu_G \hookrightarrow \ell_2) \;\geq\; \frac{n-1}{n}\cdot\frac{\gamma_G}{d}\cdot\operatorname{avg}(\mu_G^2)\,.$$

In particular,

Corollary 3.3. *Let $\{G_n\}$ be a family of regular graphs; assume G_n has n vertices and degree d_n. Suppose the normalized spectral gaps γ_{G_n}/d_n are bounded away from zero and $\operatorname{avg}(\mu_{G_n}^2) = \Omega(\log^2 n)$. Then the distance metric for this family of graphs is hard.*

In what follows we shall deal with families of graphs for which $\operatorname{avg}(\mu_G^2) = \Omega(\operatorname{Diam}(G)^2)$. We note that, in particular, any vertex-transitive graph has this property. A graph is *vertex-transitive* if all of its vertices are equivalent under automorphisms.

Proposition 3.4. *If G is a vertex-transitive graph then* $\operatorname{avg}(\mu_G^2) \geq \operatorname{Diam}(G)^2/8$.

Proof. Indeed, let r be the smallest radius such that the corresponding r-ball in μ_G contains more than $n/2$ vertices. Clearly, $\operatorname{avg}(\mu_G^2) \geq r^2/2$, while $\operatorname{Diam}(G) \leq 2r$. □

Therefore, for vertex-transitive graphs, it suffices to ensure a constant normalized spectral gap and an $\Omega(\log n)$ lower bound on the diameter.

Turning to Cayley graphs, it is well known that for (some) classes of non-Abelian groups, there exist Cayley graphs with a bounded number of generators, and a constant spectral gap (see, e. g., [8], the section on Cayley expander graphs). Since the constant number of generators guarantees that the diameter is $\Omega(\log n)$, this yields a graph as required in Corollary 3.3. (This is precisely the construction used in [11, 3]). For Abelian groups such construction is impossible, since in order to ensure a constant normalized gap γ_G/d, the number of generators must be at least $\Omega(\log n)$ (see, e. g., [8]). This might seem to be a problem, since, at least for general groups, that many generators may well cause the diameter to be $O(\log n/\log\log n) = o(\log n)$. For Abelian groups, however, this does not happen! While the following simple fact is well known (see, e. g., [8], proof of Prop. 11.5), it has apparently been overlooked in the context of hard metrics.

Let $h(p) = -p\log_2 p-(1-p)\log_2(1-p)$ be the binary entropy function. For an Abelian group H, a set $A \subseteq H$ is called symmetric if $A = -A$ (we use the additive notation for the Abelian group operation).

Proposition 3.5. *Let H be an Abelian group of order n, and let $A \subset H$ be a symmetric set of generators of size $d = c_0 \log_2 n$. Then, for any constant c_1 such that $(c_0 + c_1)\cdot h(c_1/(c_0 + c_1)) < 1$, the diameter of the corresponding Cayley graph $G(H,A)$ is $\geq c_1 \log_2 n$ for a large enough n.*

The proposition follows from the observation that the number of distinct endpoints of paths of length l in G starting at any (fixed) vertex is at most $\binom{d+l}{l}$, since due to commutativity of G it is at most the number of partitions of a set of l identical elements to d (distinct) parts. Therefore, the number of points reachable by a path of length $\leq c_1 \log_2 n$ from a fixed vertex is at most

$$\sum_{l=0}^{c_1\log_2 n} \binom{c_0\log_2 n + l}{l} \;\leq\; 2^{h\left(\frac{c_1}{c_0+c_1}\right)\cdot(c_0+c_1)\cdot\log_2 n} \;=$$

$$n^{(c_0+c_1)\cdot h\left(\frac{c_1}{c_0+c_1}\right)} \;<\; n.$$

Thus, as long as the number of generators is $O(\log n)$, our only concern is getting a constant normalized spectral gap γ_G/d. This is summed up in the following theorem.

Theorem 3.6. *Let us fix an arbitrary constant $c_0 > 0$. Let H be an Abelian group of order n, let $A \subset H$ be a symmetric set of generators of size $d = c_0 \log_2 n$ and let $G(H,A)$ be the corresponding Cayley graph. If the normalized spectral gap $\gamma_G/|A| = \Omega(1)$, then μ_G is a hard metric.*

It is well known that random choice achieves constant spectral gap (see, e. g., [1], in particular the section on Abelian groups):

Proposition 3.7. *Let H be an Abelian group of order n, and let $A \subset H$ be a random symmetric set of generators of size $d = c_0 \log_2 n$ for a suitable universal constant c_0 (100 would certainly suffice). Then, the corresponding Cayley graph $G(H,A)$ almost surely has a normalized spectral gap ≥ 0.5.*

For an efficient deterministic construction of such sets A (for any group, not only Abelian groups) see [15, Sec. 5]. Combining Theorem 3.6 and Proposition 3.7, we arrive at the main result of this section:

Theorem 3.8. *Let $G = G(H,A)$ be a Cayley graph obtained by taking a random symmetric set of generators $A \subset H$ of size $d = c_0 \log_2 |H|$ for a suitable universal constant c_0. Then, the shortest-path metric of G is almost surely a hard metric.*

Remark: Using the linear projective form

$$F_1(\Delta) \;=\; \frac{\sum_{(i,j)\in E(G)} \Delta(i,j)}{\sum_{i<j\in V(G)} \Delta(i,j)}$$

instead of $F(\Delta)$, implies the following for negative type metrics,

$$\mathrm{dist}(\mu_G \hookrightarrow \mathrm{NEG}) \;\geq\; \min_{\delta\in\mathrm{NEG}} F_1(\delta)/F_1(\mu_G)\,.$$

Arguing along the same lines as for Euclidean metrics (and recalling that a metric in NEG is a square of an ℓ_2 metric), it can be seen that the metrics of Theorem 3.8 are hard with respect to NEG as well.

4 When the Group is $\mathbb{Z}_2^n$

In this case the group is just an n-dimensional vector space over $\mathbb{Z}_2$. Any set of generators (vectors) A is automatically symmetric. Following the requirements of Corollary 3.3, we have to ensure two conditions: a constant normalized spectral gap and $\Omega(n)$ diameter.

The construction is based on good linear codes. Let $\mathcal{C} \subset \mathbb{Z}_2^m$ be a linear code (subspace) of dimension n. The *weight* $w(v)$ of a vector v is the number of nonzero entries of v. The *distance* $D(\mathcal{C})$ of $\mathcal{C}$ is the minimum weight of nonzero vectors in $\mathcal{C}$. $\mathcal{C}$ is said to be of linear distance if $D(\mathcal{C}) = \Omega(m)$. In addition, if $n = \Omega(m)$ the code is said to have a *constant rate*.

Let M be an $n\times m$ matrix whose rows form a basis for $\mathcal{C}$ (such an M is called the generator matrix of $\mathcal{C}$).

Proposition 4.1. *Let $\mathcal{C} \subset \mathbb{Z}_2^m$ be a linear code. Let M be the corresponding $n\times m$ matrix and A the set of columns of M as above. Then the spectral gap of the Cayley graph $G = G(\mathbb{Z}_2^n, A)$ is $\gamma_G = 2D(\mathcal{C})$.*

It follows that the normalized spectral gap γ_G/m is bounded away from zero if and only if $\mathcal{C}$ is a code of linear distance.

The proposition is folklore (see e. g. [1], proof of Proposition 2). Here is a sketch of the proof.

Proof. The characters $\{\chi_u\}$ of $\mathbb{Z}_2^n$, indexed by the group elements $u \in \mathbb{Z}_2^n$, are of the form

$$\chi_u(x) \;=\; (-1)^{\langle u,x\rangle}\,,$$

where the inner product is mod 2. Let $A \subset \mathbb{Z}_2^n$, $|A| = m$, be a set of generators (vectors), and let M_A be an $n\times m$ matrix over $\mathbb{Z}_2$ whose columns are the vectors of A. Keeping in mind that the eigenvectors of $G(\mathbb{Z}_2^n, A)$ are the characters, we conclude that the second largest eigenvalue λ_G of $G(\mathbb{Z}_2^n, A)$ is

$$\lambda_G \;=\; \max_{u\in\mathbb{Z}_2^n\setminus\{0\}} \sum_{a\in A} (-1)^{\langle u,a\rangle} \;=\; \max_{u\in\mathbb{Z}_2^n\setminus\{0\}} \left\{m - 2w(u^T M_A)\right\}\,.$$

Let $\mathcal{C} \subseteq \mathbb{Z}_2^n$ be a linear code generated by M_A, that is, all linear combinations of the rows of M_A. Then $\mathcal{C} = \{u^T M_A\}_{u \in \mathbb{Z}_2^n} \subset \mathbb{Z}_2^m$ and hence $\lambda_G = m - 2D(\mathcal{C})$. Since $\gamma_G = m - \lambda_G$, it follows that $\gamma_G = 2D(\mathcal{C})$. Therefore, $\gamma_G = \Omega(m)$ if and only $\mathcal{C}$ is a linear code of linear distance. □

It remains to ensure that the diameter of $G(\mathbb{Z}_2^n, A)$ is $\Omega(n)$. By Proposition 3.5, this condition will necessarily hold provided $m = O(n)$, that is, if $\mathcal{C}$ is of constant rate. We thus proved the following.

Theorem 4.2. *Let $\mathcal{C}$ be a linear code of constant rate and linear distance, and* $\dim(\mathcal{C}) = n$. *Let M be an $n \times m$ matrix whose rows form a basis for $\mathcal{C}$, and let $A \subset \mathbb{Z}_2^n$ be the set of columns of M. Then the metric of $G(\mathbb{Z}_2^n, A)$ is hard.*

Linear codes of constant rate and linear distance have received considerable attention. Their existence has been established by numerous randomized and deterministic efficient constructions, with the first explicit construction due to Justesen [9] (cf. [12]).

We conclude this section with a comparison of the construction of hard metrics due to Khot and Naor [10] and our construction. Let $\mathcal{C} \subset \mathbb{Z}_2^m$ be a linear code of constant rate and linear distance, of dimension n. Let $\mathcal{C}^\perp$ be the dual code, i. e., $\mathcal{C}^\perp = \{u \mid Mu = 0\}$ where M is the generator matrix of $\mathcal{C}$. Define an equivalence relation on $\mathbb{Z}_2^m$ by $x \equiv y$ if $(x - y) \in \mathcal{C}^\perp$. Now, let X be a quotient metric space of $\mathbb{Z}_2^m$ equipped with the Hamming metric, with respect to $\equiv$. That is, the distance between two points a and b in X is the Hamming distance between the two corresponding cosets $A, B \subset \mathbb{Z}_2^m$. Khot and Naor show that X with the induced metric is hard.

Proposition 4.3. *The above construction is isometric to the construction described in Theorem 4.2.*

Proof. Let M be a matrix as in Theorem 4.2. Then X can be viewed as the image of $\mathbb{Z}_2^m$ under the linear mapping $\phi : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$, $\phi(x) = Mx$. Define the *edges* of X as the images of Hamming edges of $\mathbb{Z}_2^m$ under ϕ. Clearly, the quotient metric of X is precisely the shortest-path metric of the resulting graph. The images of the Hamming edges are, however, precisely the column vectors of M, and the isometry follows. □

5 Additional Remarks

The constructions of Cayley graphs with hard shortest-path metric as described in Theorem 3.8 and Theorem 4.2, yield graphs of degree logarithmic in the number of vertices. It is natural to ask whether this must hold for all Cayley graphs of Abelian groups that induce a hard metric. Here we partially answer this question and show that the degree can be anything between $\Omega(\log n)$ and $O(n^{1-\varepsilon})$ for any fixed $1 > \varepsilon > 0$.

We start with the following simple fact:

Proposition 5.1. *Let H be an Abelian group, and let $m < |H|$ be a natural number. Then, there exists a symmetric set $B \subseteq H$ of size $\Theta(m)$ such that for every natural r the size of $rB = \{\sum_{i=1}^{r} b_i \mid b_i \in B\}$ is at most $r \cdot |B|$.*

Proof. Any finite Abelian group is a direct product of cyclic groups. Let $H = C_1 \times C_2 \times \ldots \times C_t$, and assume that $s_j = |C_1| \cdot |C_2| \cdot \ldots \cdot |C_j| < m$, while $s_{j+1} = |C_1| \cdot |C_2| \cdot \ldots \cdot |C_j| \cdot |C_{j+1}| \geq m$. Let a be a generator of C_{j+1}. Define $K_{j+1} = \{i \cdot a \,|\, i \in [-k,k] \subset \mathbb{N}\}$ where k is the smallest natural number such that $s_j \cdot (2k+1) \geq m$. Finally, define $B = C_1 \times C_2 \times \ldots \times C_j \times K_{j+1} \times \{0\} \times \{0\} \times \ldots$ It is easy to verify that B has the required properties. □

Theorem 5.2. *Let H be an Abelian group of order n, and let $1 > \varepsilon > 0$ be fixed. Then there exists a symmetric set of generators, A, of size $\Theta(n^{1-\varepsilon})$, such that the metric μ_G of the Cayley graph $G = G(H,A)$ is hard.*

Proof. Let $G' = G'(H,A)$ be a Cayley graph as in Theorem 3.8 (or Theorem 4.2), with $|A| = c_0 \log_2 n$. Assume for simplicity that A is augmented by $\{0\}$. Let $B \subseteq H$ be as in Proposition 5.1 with $m = n^{1-\varepsilon}$. We claim that the Cayley graph $G = G(H, A \cup B)$ has the desired properties. To see that, we employ the Poincaré form $F'(\Delta)$ similar to $F(\Delta)$ of (3.1), where in the numerator we use the edges of G' instead of the edges of G. Arguing as in Proposition 3.2, we conclude that

$$\mathrm{dist}^2(\mu_G \hookrightarrow \ell_2) \;\geq\; \frac{n-1}{n} \cdot \frac{\gamma_{G'}}{d'} \cdot \mathrm{avg}(\mu_G^2)\,. \tag{5.1}$$

We already know that the normalized spectral gap of G' is constant. Thus, it will suffice to show that the diameter of G is logarithmic. A closer examination of the proof of Proposition 3.5 reveals that the number of distinct endpoints of paths of length $\leq c_{\varepsilon/2} \log_2 n$ starting at a fixed vertex in G', is at most $n^{\varepsilon/2+o(1)}$, provided that $(c_0 + c_{\varepsilon/2}) \cdot h(c_{\varepsilon/2}/(c_0 + c_{\varepsilon/2})) \leq \varepsilon/2$. Therefore, the number of points reachable by a path of length at most $r = c_{\varepsilon/2} \log_2 n$ in G is

$$|r \cdot \{A \cup B\}| \;\leq\; |rA| \cdot |rB| \;\leq\; |rA| \cdot r|B| \;\leq\; n^{\varepsilon/2} \cdot n^{1-\varepsilon} \cdot \Theta(n^{o(1)}) \;<\; n\,.$$

(*At most* r rather than *exactly* r since both A and B contain 0.) Thus, the diameter of G is at least $c_{\varepsilon/2} \log_2 n$. This concludes the proof. □

The last issue we would like to address in this paper is the following. As the proof of Theorem 5.2 shows, the hardness of μ_G can be deduced from the hardness of $\mu_{G'}$, where G' is a sparse subgraph of G. It is natural to ask whether the hardness of μ_G itself, where G is constructed as in Theorem 3.8, can also be traced to a simple hard subgraph G' of G. What is the "core" of the hardness? It turns out that G does indeed contain such a subgraph! To avoid technicalities, we bring here only a broad outline of the argument.

First, a purely graph-theoretic argument implies the following.

Proposition 5.3. *G contains as a subgraph an expander of a bounded degree (say ≤ 500) and size $\Omega(n)$.*

Indeed, by the Cheeger Inequality for graphs (cf. [8]), which relates the normalized spectral gap to the normalized edge-expansion, G has edge expansion $\geq d/4$ where d is the degree of G. Choosing each edge of G independently at random with probability $100/d$, we obtain a graph $\tilde{G}$ which almost has the required properties. Recall that d is large enough ($\approx 100 \log n$), and hence, by a Chernoff bound, almost surely all sufficiently large subsets of vertices S, say $|S| \geq n/4$, will have an edge-boundary of size at least $c_2 \cdot 100 \cdot |S|$ in $\tilde{G}$, for some constant c_2 (e. g., $c_2 = 1/8$ suffices). Let D be the set of all vertices of degree more than 500. By Chernoff, this set is of size at most $10^{-100} \cdot n$ and has edge boundary of size at most $10^{-50}n$. Thus, in $G'' = \tilde{G} - D$ all sets S of vertices, of size, say $|S| \geq n/3$, will have an edge-boundary of size at least $c_3 \cdot 100 \cdot |S|$, where c_3 could be taken to be, e. g., $1/9$. Next, remove one by one the subsets of vertices U that have fewer than $c_3 \cdot 100 \cdot |S|$ outgoing edges in the remaining part of the graph. Since the union W of all removed has fewer than $c_3|W|$ outgoing edges, we conclude that the size of W is at most $n/3$. Thus, the graph $G' = G'' \setminus W$ is almost surely a subgraph of G of size $\geq n/2$, degree ≤ 500, and edge expansion $\geq c_3 \cdot 100$. Of course, G' is not a Cayley graph anymore, it is not even regular.

Finally, having such a large subgraph G' in G implies the hardness of μ_G as asserted by the following proposition.

Proposition 5.4. *The existence of such G', combined with the property of G (as in Proposition 3.5), that the radius of a μ_G-ball of size $n^{\Omega(1)}$ in G is at least $\Omega(\log n)$, implies the hardness of μ_G.*

Indeed, let μ' denote the restriction of μ_G to $V(G')$. The hardness of μ' can be proved by employing the Poincaré form $F_{G'}(\Delta)$ as in Equation (3.1), and using the expansion of G' to get, via the Cheeger Inequality for graphs, a lower bound on the first eigenvalue of the Laplacian of G'.

Now, using the same form $F_{G'}(\Delta)$ as in Equation (3.1), this time for μ_G, we conclude that the square of the distortion of μ_G is at least $\mathrm{dist}^2(\mu_{G'} \hookrightarrow \ell_2) \cdot \frac{\mathrm{avg}(\mu_G^2)}{\mathrm{avg}(\mu_{G'}^2)}$. Since both $\mathrm{avg}(\mu_G^2), \mathrm{avg}(\mu_{G'}^2)$ are $\Theta(\log^2 n)$, the hardness of μ_G follows. □

We end with the following open problem concerning hard metrics. In all previous constructions, as well as in the current ones, the metrics that are constructed are hard with respect to NEG and hence with respect to ℓ_1 and ℓ_2. Is there a family of hard metrics that is hard with respect to ℓ_2 but not with respect to NEG ?

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AUTHORS

Ilan Newman
Department of Computer Science
Haifa University, Haifa, 31905, Israel
ilan@cs.haifa.ac.il
http://cs.haifa.ac.il/~ilan

Yuri Rabinovich
Department of Computer Science
Haifa University, Haifa, 31905, Israel
yuri@cs.haifa.ac.il
http://cs.haifa.ac.il/~yuri

ABOUT THE AUTHORS

ILAN NEWMAN has been on the faculty at Haifa University since 1992. He obtained his B. S. and M. S. at the Technion. He received his Ph. D. in 1992 at the Computer Science Department of the Hebrew University under the supervision of Avi Wigderson. The title of his thesis was "On the complexity of simple Boolean functions." His current research interests are in finite metric embeddings, combinatorial algorithms, property testing, and computational complexity.

Born in Odessa, Ukraine, YURI RABINOVICH received his Ph. D. in 1993 from the Hebrew University under the supervision of Avi Wigderson and Nati Linial. The title of his thesis was "Theoretical properties of genetic algorithms." Currently he works at the Computer Science Department of Haifa University. He is interested in geometric phenomena in discrete structures in general, and in the theory of finite metric spaces in particular.